<u>An area problem revisited</u>

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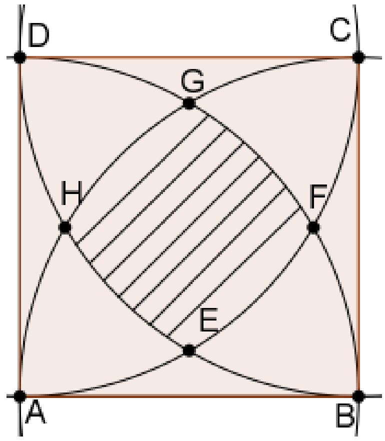
Question

Given a square ABCD with 1 as length of each side.

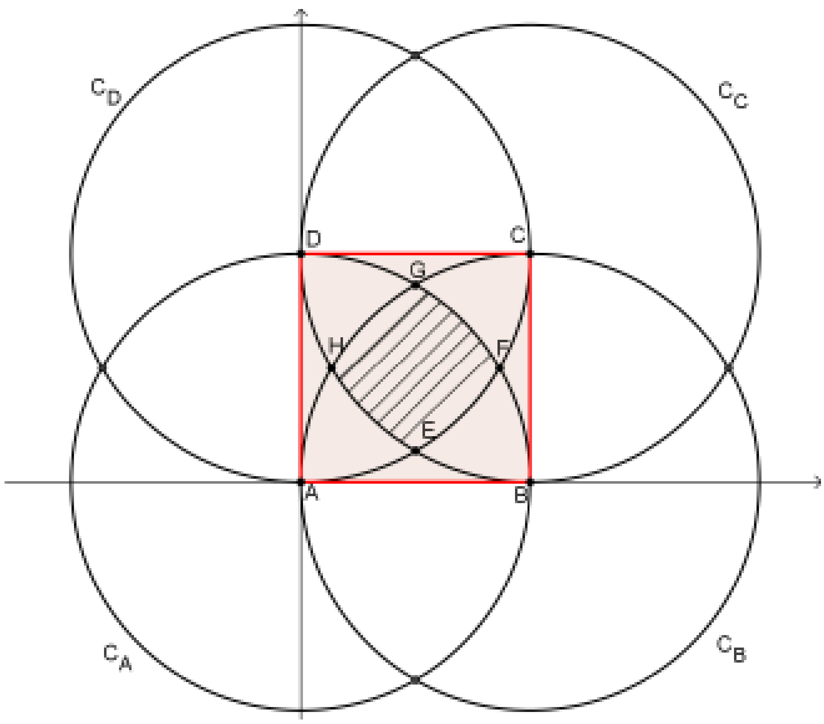
Find the area enclosed by arcs $\overset{\frown}{EF}$, $\overset{\frown}{FG}$, $\overset{\frown}{GH}$ and $\overset{\frown}{HE}$.



There are many ways to solve this problem.
Here is a way in which **definite integration** is employed.

Solution

Place the square in the grid.
A(0, 0), B(1, 0), C(1, 1), D(0,1) .



There are 4 circles as in the diagram.

The equations of the 4 arcs of circles lying **inside the given square** where $0 \le x \le 1$, are shown below:

Centre A, $C_A: \quad y = \sqrt{1-x^2}$

Centre B, $C_B: \quad y = \sqrt{1-(1-x)^2}$

Centre C,

$C_C: \quad y = 1-\sqrt{1-(1-x)^2}$

Centre D, $C_D: \quad y = 1-\sqrt{1-x^2}$

Solving C_B and C_C, we can get the point $H = \left(1-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Solving C_A and C_B, we can get the point $G = \left(\frac{1}{2}, 1-\frac{\sqrt{3}}{2}\right)$.

Required area = S

= 2 × [area under the curve HG in circle C_B – area under the curve HE of circle C_C]

$$=2\int_{1-\frac{\sqrt{3}}{2}}^{\frac{1}{2}}\left[\sqrt{1-(1-x)^2}-\left(1-\sqrt{1-(1-x)^2}\right)\right]dx$$

$$=2\int_{1-\frac{\sqrt{3}}{2}}^{\frac{1}{2}}\left[2\sqrt{1-(1-x)^2}-1\right]dx$$

Now, for $I=\int\sqrt{1-(1-x)^2}\,dx$

Put $1-x=\sin\theta,\quad dx=-\cos\theta d\theta$

$$\sqrt{1-(1-x)^2}=\sqrt{1-\sin^2\theta}=\cos\theta$$

$$\therefore I=\int\cos\theta[-\cos\theta d\theta]=-\frac{1}{2}\int(1+\cos 2\theta)d\theta=-\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]+c=-\frac{1}{2}\left[\sin^{-1}(1-x)+\sin\theta\cos\theta\right]+c$$

$$=-\frac{1}{2}\left[\sin^{-1}(1-x)+(1-x)\sqrt{1-(1-x)^2}\right]+c$$

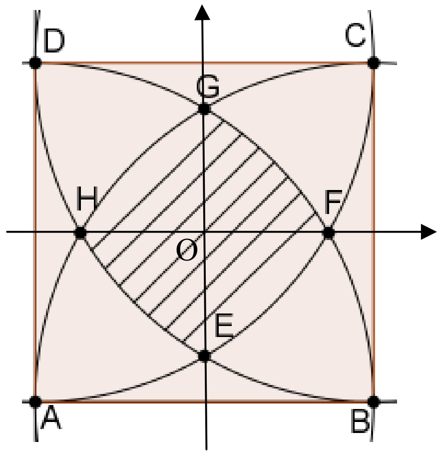
$$\therefore S=2\int_{1-\frac{\sqrt{3}}{2}}^{\frac{1}{2}}\left[2\sqrt{1-(1-x)^2}-1\right]dx=\left[-2\left[\sin^{-1}(1-x)+(1-x)\sqrt{1-(1-x)^2}\right]-2x\right]_{1-\frac{\sqrt{3}}{2}}^{\frac{1}{2}}$$

$$=\left\{-2\left[\sin^{-1}\frac{1}{2}+\frac{1}{2}\sqrt{1-\left(\frac{1}{2}\right)^2}\right]-2\left(\frac{1}{2}\right)\right\}-\left\{-2\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}\right]-2\left(1-\frac{\sqrt{3}}{2}\right)\right\}$$

$$=\left\{-2\left[\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right]-1\right\}-\left\{-2\left[\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right]-2\left(1-\frac{\sqrt{3}}{2}\right)\right\}$$

$$=1-\sqrt{3}+\frac{\pi}{3}$$

A better way to use calculus is to use parametric integration.
Place the axes and origin O as in the figure in the left.
We like to find the shaded area OFG in the first quadrant and multiply by 4.



The arc FG in fact is part of the circle centre A, radius AB.
The Cartesian equation for this circle is

$$\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=1$$

The parametric equation is therefore:

$$\begin{cases} x=\cos t-\frac{1}{2} \\ y=\sin t-\frac{1}{2} \end{cases}$$

Point B is where $t = 0$, Point F is at $t=\frac{\pi}{6}$. Point G is at $t=\frac{\pi}{3}$.

Therefore required area is

$$S=4\int_{\pi/6}^{\pi/3} y\,dx=-4\int_{\pi/6}^{\pi/3} y\frac{dx}{dt}dt=4\int_{\pi/6}^{\pi/3}\left(\sin t-\frac{1}{2}\right)d\left(\cos t-\frac{1}{2}\right)$$

$$=-4\int_{\pi/6}^{\pi/3}\left(\sin t-\frac{1}{2}\right)(-\sin t)dt=4\int_{\pi/6}^{\pi/3}\left(\sin^2 t-\frac{1}{2}\sin t\right)dt=\underline{\underline{1-\sqrt{3}+\frac{\pi}{3}}}$$